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REDUCTION OF THE FRACTIONAL DIFFERENTIAL MODEL OF FLAT-RADIAL FLUID FLOW IN A FRACTAL MEDIUM TO A SECOND-KIND FREDHOLM EQUATION

Aliyev N.A.[1], Rasulov M.A.[2*], Jalalov G.I.[2], Sinsoysal B.[3], Xanbabayeva M.H.[2]

[1]Baku State University, Faculty of Applied Mathematics and Cybernetics, Azerbaijan
23, acad. Zahid Khalilov str., Baku AZ1148
[2]Ministry of Science and Education of the Republic of Azerbaijan,
Institute of Oil and Gas, Azerbaijan,
9, Fikret Amirov str., Baku, AZ1000
[3]Istanbul Beykent University, Faculty of Engineering and Architecture,
Department of Computer Engineering, Turkey
34396, Sariyer, Istanbul
**Corresponding author: mresulov@gmail.com*

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Summary. Over the last decade, interest in the theory of mass transfer in fractal media stems from geophysical and experimental data demonstrating the non-linearity of porous media, including their block structure and fractal properties. In such media, the filtration process, even for homogeneous mixtures, is modelled by non-integer-order differential equations, complicating theoretical research. In this paper, a solution is constructed for the fractional-order non-stationary diffusion equation describing the plane-radial motion of single-phase fluids in fractal porous media. To solve the reduced spectral problem for the derivative of the Laplace transform function, a Volterra integral equation of the second kind with a regular kernel is derived. For this integral equation, a resolvent in the form of a rapidly converging series is obtained by successive substitution. A necessary condition for the convergence of the series is found, and an analytical expression for the derivative of the Laplace transform of the desired pressure function is obtained. By integrating the resulting analytical expression for the Laplace transform with respect to the spatial variable, an integral representation is found. Finally, by applying the inverse Laplace transform, an expression for the dynamic distribution of the desired pressure function is obtained. The resulting expression, firstly, allows one to study the problem of unsteady fluid flow in a radial medium of fractal nature, and secondly, also answers the question of the existence and uniqueness of the problem under consideration. Using numerical integration methods, one can calculate the values of the pressure function at any point *r* and any time *t*, with an error estimate. Furthermore, the obtained results allow one to evaluate the solution found using approximate methods.

Introduction

In many fields of science and technology, the application of the theory of fractals, which describes the laws of irregular heterogeneous environments new research directions have begun to intensively develop. Many scientific sources in the literature are devoted to the statistical classification of porous media with fractal character. In this classification, either the pore space, or the rock (collector), or the surface of the pore system, or the volume of the liquid filling the porous medium can be considered fractal (Afonin, Sukhinov, 2009; Barabanov, 2016; Belevtsev, Lukashchuk, 2020; Gazizov, Lukashchuk, 2017; Nigmatullin, 1986), laboratory experiments show that the sand-clay medium has a fractal nature and the fractal dimension can vary from 2.6 to 2.8.

The equations involved in this process are described by means of a fractional form ordinary or partial differential equation or a system of equations. In addition to being an unusual differential equation, fractional differential equations reveal an unexpected simplicity in revealing the essence of many natural phenomena – complex systems found in modern physics, mechanics, biology, chemistry, engineering, etc.

Fractional differential equations are not only a unique type of differential equation, but they also exhibit a surprising simplicity in capturing the essence of many natural phenomena including intricate systems found in contemporary physics, mechanics, biology, chemistry, engineering, and other fields. In this regard, fractional derivative and differential calculus has recently become a rapidly developing field of mathematical analysis.

R.R.Nigmatullin showed that the diffusion process in media with a "Koch tree" type fractal structure is described by a generalised

$$\frac{\partial^\alpha c_1(x,t)}{\partial t^\alpha} = D_1 \frac{\partial^2 c_1(x,t)}{\partial x^2}$$

equation of motion in special derivatives, and this model can serve as a model of a porous medium in which the diffusion process occurs. Here, D_1 is a positive constant characterising the flow and the medium. From this model, a fractional derivative relaxation equation related to the fractal dimension of the medium can also be obtained (Afonin, Sukhinov, 2009; Mirzadzhanzade et.al., 2004).

As it is known in the literature, this aspect, whose geometric structure retains its basic properties while considering at different length scales, i.e., it has the property of self-similarity, has found application in the geosciences including the solution of oilfield development problems, and development work continues. The application of models expressed by fractional order equations allows the development of general methods modeling flow processes in porous media with complex inhomogeneous permeability and facilitates the consideration of the influence of physical processes occurring in these media.

The filtration process in fractured media usually occurs under anomalous kinetic conditions, i.e. it does not obey normal Gaussian statistics. The presence of cracks in a porous medium creates a nonlocal effect due to spatial variation. Sometimes cracks are formed as a result of explosions in the beds and cause severe inhomogeneities (Mirzadzhanzade et.al., 2004).

In the case of fractal-like fractures around the well, fluid filtration can be accomplished not by classical Darcy's law but by means of a modified, i.e. fractional order derivative.

In the case of radial flow, the modified Darcy's law is expressed by

$$w = -\frac{k_\nu}{\mu} D^\nu_{r_w+} p(r,t), \ \nu \in (0,1), \qquad (1)$$

(Barabanov, 2016; Gazizov, Lukashchuk, 2017). Here w is the filtration velocity, p is the pressure, k_ν is the fractional analog of permeability, μ is the viscosity of the fluid, r is the distance from the center of the well, t is the time variable and

$$D^\nu_{r_w+} p(r,t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dr} \int_{r_w}^{r} \frac{p(\rho,t)d\rho}{(r-\rho)^\nu},$$

(Kilbas et al., 2006; Oldham, Spanier, 1974; Podlubny, 1999; Samko et al., 1993).

If we substitute expression (1) into the classical

$$\frac{\partial(\phi\rho)}{\partial t} + \text{div}\,(\rho w) = q,$$

continuity equation, we get

$$\frac{\partial p(r,t)}{\partial t} = a_\nu \frac{1}{r} \frac{\partial}{\partial r}\left[r\, D^\nu_{r_w+} p(r,t)\right] + q,$$
$$0 < r_w < r < r_c, \ t > 0. \qquad (2)$$

Here, r_w and r_c are constants that describe the contour of the well and the bed in the reservoir, respectively. $a_\nu = \frac{k_\nu}{\mu\,\phi c_f}$ is a positive constant coefficient, ϕ is the porosity, q is the source function, and $\rho = \rho(p)$ is the classical equation of state. Previously, let us first consider the case where k_ν, μ and ϕ are constant.

Now, let's investigate the solution of equation (2) under the following conditions

$$p(r,0) = p_0(r), \qquad (3)$$

$$p(r_w,t) = p_w(t), \ p(r_c,t) = p_c(t), \qquad (4)$$

where, $p_0(r)$, $p_w(t)$ and $p_c(t)$ are known continuous functions. Equation (2) was studied in (Aliyev et al., 2023) in the case of rectilinear fluid flow in a porous medium with fractal nature and when the pressure function p does not depend on t (Suleymanov et al., 2005).

Converting the differential problem into the integral equation

It can be expressed equation (2) as

$$D_t p(r,t) = \frac{a_\nu}{r}\left(rD\, D^\nu_{r_w+} p(r,t) + \ D^\nu_{r_w+} p(r,t)\right) + q,$$
$$t \in (0,\infty), r \in (r_w, r_c) \qquad (5)$$

using the notation $D. = \frac{\partial .}{\partial r}$. Applying the Laplace transform

$$\tilde{p}(r,\lambda)=\int_0^\infty e^{-\lambda t}p(r,t)dt$$

to both sides of equation (5), we get

$$D_{r_w+}^{1+\nu}\tilde{p}(r,\lambda)+\frac{1}{r}D_{r_w+}^{\nu}\tilde{p}(r,\lambda)-\frac{\lambda}{a_\nu}\tilde{p}(r,\lambda)=-F(r,\lambda), \quad (6)$$

where

$$F(r,\lambda)=\frac{p_0(r)+\tilde{q}(r,\lambda)}{a_\nu},$$

$$D_{r_w+}^{\nu}\tilde{p}(r,\lambda)=\frac{1}{\Gamma(1-\nu)}\frac{d}{dr}\int_{r_w}^{r}\frac{\tilde{p}(\rho,\lambda)d\rho}{(r-\rho)^{\nu}}. \quad (7)$$

If we apply operator

$$I_{r_w+}^{\nu}(\cdot)=\frac{1}{\Gamma(\nu)}\int_{r_w}^{r}\frac{(\cdot)}{(r-\rho)^{1-\nu}}d\rho \quad (8)$$

to both sides of equation (6), we obtain

$$D_{r_w+}^{1}\tilde{p}(r,\lambda)+I_{r_w+}^{\nu}\left(\frac{1}{r}D_{r_w+}^{\nu}\tilde{p}(r,\lambda)\right)-$$
$$-\frac{\lambda}{a_\nu}I_{r_w+}^{\nu}\left(\tilde{p}(r,\lambda)\right)+C_1\frac{r^{\nu-1}}{(\nu-1)!}=$$
$$=-I_{r_w+}^{\nu}\left(F(r,\lambda)\right)+D_{r_w+}^{1}\widetilde{p'}(r,\lambda)\Big|_{r=r_w}. \quad (9)$$

Using the relation

$$\int_{r_w}^{r}\frac{(r-\rho)^{-\nu}}{(-\nu)!}\tilde{p}(\rho,\lambda)d\rho=\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{1-\nu}}{(1-\nu)!}+$$
$$+\int_{r_w}^{r}\frac{(r-\rho)^{1-\nu}}{(1-\nu)!}\widetilde{p'}(\rho,\lambda)d\rho,$$

we can express (7) as

$$D_{r_w+}^{\nu}\tilde{p}(r,\lambda)=\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{-\nu}}{(-\nu)!}+$$
$$+\int_{r_w}^{r}\frac{(r-\rho)^{-\nu}}{(-\nu)!}\widetilde{p'}(\rho,\lambda)d\rho.$$

In this instance, (9) takes on the following form

$$D_{r_w+}^{1}\tilde{p}(r,\lambda)+I_{r_w+}^{\nu}\left(\frac{1}{r}\int_{r_w}^{r}\frac{(r-\rho)^{-\nu}}{(-\nu)!}\widetilde{p'}(\rho,\lambda)d\rho\right)-$$
$$-\frac{\lambda}{a_\nu}I_{r_w+}^{\nu}\left(\tilde{p}(r,\lambda)\right)=-C_1\frac{r^{\nu-1}}{(\nu-1)!}-$$
$$-I_{r_w+}^{\nu}\left(\frac{1}{r}\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{-\nu}}{(-\nu)!}\right)-$$
$$-I_{r_w+}^{\nu}\left(F(r,\lambda)\right)+D_{r_w+}^{1}\widetilde{p'}(r,\lambda)\Big|_{r=r_w}. \quad (10)$$

Let's consider the

$$J_1=I_{r_w+}^{\nu}\left(\frac{1}{r}\int_{r_w}^{r}\frac{(r-\rho)^{-\nu}}{(-\nu)!}\widetilde{p'}(\rho,\lambda)d\rho\right),$$
$$J_2=\frac{\lambda}{a_\nu}I_{r_w+}^{\nu}\left(\tilde{p}(r,\lambda)\right)$$

expressions included in the equation (10). According to the definition of the integral, we can write J_1 as

$$J_1=I_{r_w+}^{\nu}\left(\frac{1}{r}\int_{r_w}^{r}\frac{(r-\rho)^{-\nu}}{(-\nu)!}\widetilde{p'}(\rho,\lambda)d\rho\right)=$$
$$=\int_{r_w}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}d\eta\frac{1}{\eta}\int_{r_w}^{\eta}\frac{(\eta-\rho)^{-\nu}}{(-\nu)!}\widetilde{p'}(\rho,\lambda)d\rho=$$
$$=\int_{r_w}^{r}\widetilde{p'}(\rho,\lambda)d\rho\int_{\rho}^{r}\frac{1}{\eta}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{(\eta-\rho)^{-\nu}}{(-\nu)!}d\eta. \quad (11)$$

Using the equation

$$\tilde{p}(\rho,\lambda)=\int_{r_w}^{\rho}\widetilde{p'}(\eta,\lambda)d\eta+\tilde{p}(r_w,\lambda),$$

for J_2 we have

$$J_2=\frac{\lambda}{a_\nu}\int_{r_w}^{r}\frac{(r-\rho)^{\nu-1}}{(\nu-1)!}d\rho\left[\int_{r_w}^{\rho}\widetilde{p'}(\eta,\lambda)d\eta+\tilde{p}(r_w,\lambda)\right]=$$
$$=\frac{\lambda}{a_\nu}\int_{r_w}^{r}\frac{(r-\rho)^{\nu-1}}{(\nu-1)!}d\rho\int_{r_w}^{\rho}\widetilde{p'}(\eta,\lambda)d\eta+$$
$$+\frac{\lambda}{a_\nu}\int_{r_w}^{r}\frac{(r-\rho)^{\nu-1}}{(\nu-1)!}\tilde{p}(r_w,\lambda)d\rho=$$
$$=\frac{\lambda}{a_\nu}\int_{r_w}^{r}\widetilde{p'}(\eta,\lambda)d\eta\int_{\eta}^{r}\frac{(r-\rho)^{\nu-1}}{(\nu-1)!}d\rho+J_3, \quad (12)$$

where

$$J_3=\frac{\lambda}{a_\nu}\int_{r_w}^{r}\frac{(r-\rho)^{\nu-1}}{(\nu-1)!}\tilde{p}(r_w,\lambda)d\rho.$$

Substituting (11) and (12) into (10) yields

$$D_{r_w+}^{1}\tilde{p}(r,\lambda)+\int_{r_w}^{r}\widetilde{p'}(\rho,\lambda)d\rho\left[\int_{\rho}^{r}\frac{1}{\eta}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{(\eta-\rho)^{-\nu}}{(-\nu)!}d\eta-\frac{\lambda}{a_\nu}\int_{\rho}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}d\eta\right]=$$

$$=-C_1\frac{r^{\nu-1}}{(\nu-1)!}+\frac{\lambda}{a_\nu}\int_{r_w}^{r}\frac{(r-\rho)^{\nu-1}}{(\nu-1)!}\tilde{p}(r_w,\lambda)d\rho-I_{r_w+}^{\nu}\left(\frac{1}{r}\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{-\nu}}{(-\nu)!}\right)-$$

$$-I_{r_w+}^{\nu}\big(F(r,\lambda)\big)+D_{r_w+}^{1}\widetilde{p'}(r,\lambda)\Big|_{r=r_w}. \tag{13}$$

Some integrals in the last equation can be calculated as follows:

$$\int_{\rho}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}d\eta=-\frac{1}{(\nu-1)!}\int_{\rho}^{r}(r-\eta)^{\nu-1}\,d(r-\eta)=\frac{(r-\rho)^{\nu}}{\nu!},$$

$$\frac{\lambda}{a_\nu}\int_{r_w}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\tilde{p}(r_w,\lambda)\,d\eta=\frac{\lambda}{a_\nu}\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{\nu}}{\nu!},$$

$$J_4=I_{r_w+}^{\nu}\left(\frac{1}{r}\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{-\nu}}{(-\nu)!}\right)=\int_{r_w}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{1}{\eta}\tilde{p}(r_w,\lambda)\frac{(\eta-r_w)^{-\nu}}{(-\nu)!}d\eta=$$

$$=\tilde{p}(r_w,\lambda)\int_{r_w}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{1}{\eta}\frac{(\eta-r_w)^{-\nu}}{(-\nu)!}d\eta.$$

In this case, equation (13) can be expressed as

$$D_{r_w+}^{1}\tilde{p}(r,\lambda)+\int_{r_w}^{r}\widetilde{p'}(\rho,\lambda)d\rho\left[\int_{\rho}^{r}\frac{1}{\eta}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{(\eta-\rho)^{-\nu}}{(-\nu)!}d\eta-\frac{\lambda}{a_\nu}\frac{(r-\rho)^{\nu}}{\nu!}\right]=$$

$$=-C_1\frac{r^{\nu-1}}{(\nu-1)!}+\frac{\lambda}{a_\nu}\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{\nu}}{\nu!}-\tilde{p}(r_w,\lambda)\int_{r_w}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{1}{\eta}\frac{(\eta-r_w)^{-\nu}}{(-\nu)!}d\eta-$$

$$-I_{r_w+}^{\nu}\big(F(r,\lambda)\big)+D_{r_w+}^{1}\widetilde{p'}(r,\lambda)\Big|_{r=r_w}. \tag{14}$$

Finally, denoted by

$$K(r,\rho,\lambda)=\int_{\rho}^{r}\frac{1}{\eta}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{(\eta-\rho)^{-\nu}}{(-\nu)!}d\eta-\frac{\lambda}{a_\nu}\frac{(r-\rho)^{\nu}}{\nu!}, \tag{15}$$

$$\Phi(r,\lambda)=-C_1\frac{r^{\nu-1}}{(\nu-1)!}+\Phi_1(r,\lambda), \tag{16}$$

$$\Phi_1(r,\lambda)=\frac{\lambda}{a_\nu}\tilde{p}(r_w,\lambda)\frac{(r-r_w)^{\nu}}{\nu!}-\tilde{p}(r_w,\lambda)\int_{r_w}^{r}\frac{(r-\eta)^{\nu-1}}{(\nu-1)!}\frac{1}{\eta}\frac{(\eta-r_w)^{-\nu}}{(-\nu)!}d\eta-$$

$$-I_{r_w+}^{\nu}\big(F(r,\lambda)\big)+D_{r_w+}^{1}\widetilde{p'}(r,\lambda)\Big|_{r=r_w},$$

for $\widetilde{p'}(r,\lambda)$ we obtain the following Volterra type-II integral equation

$$\widetilde{p'}(r,\lambda)+\int_{r_w}^{r}K(r,\rho,\lambda)\widetilde{p'}(\rho,\lambda)d\rho=\Phi(r,\lambda). \tag{17}$$

As can be seen from (15) and (16), for $\rho\le r$, the functions $K(r,\rho,\lambda)$ and $\Phi(r,\lambda)$ are continuous functions with respect to the variable r.

Applying the method of successive substitution to the (17) Volterra integral equation, we find that

$$\tilde{p}'(\rho,\lambda)=\Phi(\rho,\lambda)-\int_{r_w}^{\rho}K_1(\rho,\theta,\lambda)\tilde{p}'(\theta,\lambda)d\theta. \tag{18}$$

Here, $K_1(r,\rho,\lambda)=K(r,\rho,\lambda)$. Considering (18),

$$\tilde{p}'(r,\lambda)=\Phi(r,\lambda)-\int_{r_w}^{r}K_1(r,\rho,\lambda)\tilde{p}'(\rho,\lambda)d\rho=$$

$$=\Phi(r,\lambda)-\int_{r_w}^{r}K_1(r,\rho,\lambda)\left[\Phi(\rho,\lambda)-\int_{r_w}^{\rho}K_1(\rho,\theta,\lambda)\tilde{p}'(\theta,\lambda)d\theta\right]d\rho=$$

$$=\Phi(r,\lambda)-\int_{r_w}^{r}K_1(r,\rho,\lambda)\Phi(\rho,\lambda)\,d\rho+\int_{r_w}^{r}K_1(r,\rho,\lambda)d\rho\int_{r_w}^{\rho}K_1(\rho,\theta,\lambda)\tilde{p}'(\theta,\lambda)d\theta.$$

Changing the order of integration in the last expression, we get

$$\tilde{p}'(r,\lambda)=\Phi(r,\lambda)-\int_{r_w}^{r}K_1(r,\rho,\lambda)\Phi(\rho,\lambda)\,d\rho\ +\int_{r_w}^{r}K_2(r,\rho,\lambda)\tilde{p}'(\rho,\lambda)d\rho, \tag{18_1}$$

where

$$K_2(r,\theta,\lambda)=\int_{\rho}^{r}K_1(r,\rho,\lambda)K_1(\rho,\theta,\lambda)d\rho. \tag{19_1}$$

By repeating this process n times, we have

$$\tilde{p}'(r,\lambda)=\Phi(r,\lambda)-\int_{r_w}^{r}K_1(r,\rho,\lambda)\Phi(\rho,\lambda)d\rho+\int_{r_w}^{r}K_2(r,\rho,\lambda)\Phi(\rho,\lambda)d\rho-$$

$$-\int_{r_w}^{r}K_3(r,\rho,\lambda)\Phi(\rho,\lambda)d\rho+\cdots+(-1)^n\int_{r_w}^{r}K_{n-1}(r,\rho,\lambda)\Phi(\rho,\lambda)d\rho+$$

$$+\int_{r_w}^{r}K_n(r,\rho,\lambda)\tilde{p}'(\rho,\lambda)d\rho. \tag{18_n}$$

Here,

$$K_n(r,\rho,\lambda)=\int_{\rho}^{r}K_1(r,\theta,\lambda)K_{n-1}(\rho,\theta,\lambda)d\theta. \tag{19_n}$$

We can write equation (18_n) as

$$\tilde{p}'(r,\lambda) = \Phi(r,\lambda) + \sum_{m=1}^{n-1}(-1)^m \int_{r_w}^{r} K_m(r,\rho,\lambda)\,\Phi(\rho,\lambda)d\rho + \int_{r_w}^{r} K_n(r,\rho,\lambda)\tilde{p}'(\rho,\lambda)d\rho =$$

$$= \Phi(r,\lambda) + \int_{r_w}^{r}\left[\sum_{m=1}^{n}(-1)^m K_m(r,\rho,\lambda)\right]\Phi(\rho,\lambda)d\rho + \int_{r_w}^{r} K_n(r,\rho,\lambda)\tilde{p}'(\rho,\lambda)d\rho. \qquad (20)$$

Taking into account the summation $\sum_{m=1}^{n}(-1)^m K_m(r,\rho,\lambda)$, the solution of the integral equation (18) is

$$\tilde{p}'(r,\lambda) = \lim_{n\to\infty}\left\{\Phi(r,\lambda) + \int_{r_w}^{r}\left[\sum_{m=1}^{n}(-1)^m K_m(r,\rho,\lambda)\right]\Phi(\rho,\lambda)d\rho\right\} +$$

$$+ \int_{r_w}^{r} K_n(r,\rho,\lambda)\tilde{p}'(\rho,\lambda)d\rho,$$

$$\tilde{p}'(r,\lambda) = \Phi(r,\lambda) + \int_{r_w}^{r} R(r,\rho,\lambda)\,\Phi(\rho,\lambda)d\rho\,. \qquad (21)$$

Here, the expression

$$R(r,\rho,\lambda) = \sum_{m=1}^{\infty}(-1)^m K_m(r,\rho,\lambda) \qquad (22)$$

is called resolvent.

Let us now examine whether the series (22) converges. $K(r,\rho,\lambda)$ is a continuous and bounded function, i.e., for $\rho < r$ the following evaluations are valid

$$|K_1(r,\rho,\lambda)| \le M,$$

$$|K_2(r,\theta,\lambda)| = \left|\int_{\theta}^{r} K_1(r,\rho,\lambda)K_1(\rho,\theta,\lambda)d\rho\right| \le M^2(r-\theta),$$

$$|K_3(r,\theta,\lambda)| = \left|\int_{\theta}^{r} K_1(r,\theta,\lambda)K_2(\rho,\theta,\lambda)d\rho\right| \le M^3\int_{\rho}^{r}(\rho-\theta)d\rho = M^3\frac{(r-\theta)^2}{2!},$$

....

$$|K_n(r,\theta,\lambda)| = \left|\int_{\theta}^{r} K_1(r,\rho,\lambda)K_{n-1}(\rho,\theta,\lambda)d\rho\right| \le M^n\int_{\rho}^{r}\frac{(\rho-\theta)^{n-2}}{(n-2)!}d\rho = M^n\frac{(r-\theta)^{n-1}}{(n-1)!}.$$

For the upper bound of the resolvent (22) we obtain the following estimate

$$\frac{M}{(n-1)!}[M(r-\theta)]^{n-1} \le \frac{M}{(n-1)!}q^{n-1},$$

where

$$q = M(r_c - r_w).$$

Thus, the series

$$M + Mq + M\frac{q^2}{2} + \cdots + M\frac{q^{n-1}}{(n-1)!} + \cdots$$

is absolutely and regularly convergent in the triangle domain ($r_w \le r \le r_{c,}\ r_w \le \rho \le r$), for arbitrary q.

Now, to find the function $\tilde{p}(r,\lambda)$, let us integrate equation (21) with respect to r from r_w to r_c

$$\tilde{p}(r,\lambda) = \int_{r_w}^{r} \Phi(\rho,\lambda)d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\,\Phi(\eta,\lambda)d\eta\right]dr + C_2.$$

Considering (17), the last equation can be written as follows:

$$\tilde{p}(r,\lambda) = -C_1\left\{\int_{r_w}^{r}\frac{\rho^{\nu-1}}{(\nu-1)!}d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\frac{\eta^{\nu-1}}{(\nu-1)!}d\eta\right]d\rho\right\} + C_2 +$$
$$+\left\{\int_{r_w}^{r}\Phi_1(\rho,\lambda)d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\Phi_1(\eta,\lambda)\,d\eta\right]d\rho\right\}. \tag{23}$$

Using the following expressions

$$A(r) = \int_{r_w}^{r}\frac{\rho^{\nu-1}}{(\nu-1)!}d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\frac{\eta^{\nu-1}}{(\nu-1)!}d\eta\right]d\rho,$$

and

$$B(r) = \int_{r_w}^{r}\Phi_1(\rho,\lambda)d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\Phi_1(\eta,\lambda)\,d\eta\right]d\rho$$

the equation (23) can be rewritten as follows

$$C_1A(r) - C_2 = \tilde{p}'(r,\lambda) - B(r). \tag{24}$$

Using the following conditions

$$\tilde{p}(r_w,\lambda) = \tilde{p}_w(\lambda),\;\; \tilde{p}(r_c,\lambda) = \tilde{p}_c(\lambda)$$

to find the unknown constants, we get

$$\begin{cases} C_1A(r_w) - C_2 = \tilde{p}_w(\lambda) - B(r_w), \\ C_1A(r_c) - C_2 = \tilde{p}_c(\lambda) - B(r_c). \end{cases}$$

The solution of the obtained system of algebraic equations by using Cramer's method is

$$C_1 = \frac{\tilde{p}_c(\lambda) - \tilde{p}_w(\lambda) - \big(B(r_c) - B(r_w)\big)}{A(r_c)},$$
$$C_2 = \frac{-A(r_c)\big(\tilde{p}_w(\lambda) - B(r_w)\big)}{A(r_c)} = -\big(\tilde{p}_w(\lambda) - B(r_w)\big).$$

Substituting these expressions into (24), for $\tilde{p}(r,\lambda)$ we obtain the following expression

$$\tilde{p}(r,\lambda) = -\frac{\tilde{p}_c(\lambda) - \tilde{p}_w(\lambda) - \big(B(r_c) - B(r_w)\big)}{A(r_c)}\left\{\int_{r_w}^{r}\frac{\rho^{\nu-1}}{(\nu-1)!}d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\frac{\eta^{\nu-1}}{(\nu-1)!}d\eta\right]d\rho\right\} -$$
$$-\big(\tilde{p}_w(\lambda) - B(r_w)\big) + \left\{\int_{r_w}^{r}\Phi_1(\rho,\lambda)d\rho + \int_{r_w}^{r}\left[\int_{r_w}^{\rho} R(\rho,\eta,\lambda)\Phi_1(\eta,\lambda)\,d\eta\right]d\rho\right\}. \tag{25}$$

Using the obtained Laplace transform $\tilde{p}(r,\lambda)$ with the help of the inverse transform

$$p(r,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\tilde{p}(r,\lambda)d\lambda\,,\gamma=$$

$$=Re\lambda>\ \lambda_0, i=\sqrt{-1}$$

we can find the values of the original pressure function at arbitrary points r and t.

Conclusion

The fractional with 1+α (0<α<1) order diffusion equation modelling of the flat-radial flow of one-phase liquids in porous media with fractal structure is transformed into the spectral boundary value problem for derivatives with respect to the Laplace transformation.

The solution of the spectral problem reduces to the second type Volterra integral equation.

In order to find the solution of the integral equation, the method of successive substitution is used to construct a resolvent for the solution and the solution is obtained as a convergent series.

The obtained representation allows for the proof of the existence of a solution of the considered problem and discovery of its approximate solution provided that the error is evaluated.

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СВЕДЕНИЕ ЗАДАЧИ ДРОБНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ, ОПИСЫВАЮЩЕГО ПРОЦЕСС ПЛОСКО-РАДИАЛЬНОГО ДВИЖЕНИЯ ЖИДКОСТИ ВО ФРАКТАЛЬНОЙ СРЕДЕ, К ИНТЕГРАЛЬНОМУ УРАВНЕНИЮ ФРЕДГОЛЬМА ВТОРОГО РОДА

Алиев Н.А.[1], Расулов М.А.[2*], Джалалов Г.И.[2], Синсойсал Б.[3], Ханбабаева М.Х.[2]

[1]*Бакинский Государственный Университет, Факультет прикладной математики и кибернетики, Азербайджан AZ1148, Баку, ул. акад. Захида Халилова, 23*

[2]*Министерство науки и образования Азербайджанской Республики, Институт нефти и газа, Азербайджан AZ1000, Баку, ул. Фикрета Амирова, 9*

[3]*Стамбульский Университет Бейкент, Факультет Инженерии и Архитектуры, Кафедра Компьютерной Инженерии, Турция 34396, Сарыер, Стамбул*

**Автор, отвечающий за переписку: mresulov@gmail.com*

Резюме. В последнее десятилетие теория массопереноса во фрактальных средах привлекает как практический, так и теоретический интерес. Этот интерес обусловлен геофизическими и экспериментальными данными, демонстрирующими нелинейность пористых сред, включая их блочную структуру и фрактальные свойства. В таких средах процесс фильтрации даже для однородных смесей моделируется дифференциальными уравнениями нецелого порядка, что затрудняет теоретические исследования. В данной работе построено решение нестационарного уравнения диффузии дробного порядка, описывающего плоскорадиальное движение однофазных жидкостей во фрактальных пористых средах. Для решения редуцированной спектральной задачи для производной функции преобразования Лапласа выведено интегральное уравнение Вольтерры второго рода с регулярным ядром. Для этого интегрального уравнения методом последовательной подстановки получена резольвента в виде быстро сходящегося ряда. Найдено необходимое условие сходимости ряда и получено аналитическое выражение для производной функции преобразования Лапласа искомой функции давления. Интегрируя полученное аналитическое выражение для преобразования Лапласа по пространственной переменной, находим интегральное представление. Наконец, применяя обратное преобразование Лапласа, получаем выражение для динамического распределения искомой функции давления. Полученное выражение, во-первых, позволяет исследовать задачу нестационарного течения жидкости в радиальной среде фрактальной природы, а во-вторых, также отвечает на вопрос о существовании и единственности рассматриваемой задачи. Используя методы численного интегрирования, можно вычислить значения функции давления в любой точке r и в любой момент времени t с оценкой погрешности. Кроме того, полученные результаты позволяют оценить найденное решение приближенными методами.

Ключевые слова: *массоперенос во фрактальных средах, нестационарное уравнение диффузии в плоскорадиальной области, преобразование Лапласа, интегральное уравнение Вольтерра второго рода с регулярным ядром, метод последовательной подстановки*

FRAKTAL MÜHITDƏ MAYENİN MÜSTƏVİ-RADİAL HƏRƏKƏT PROSESİNİ TƏSVİR EDƏN KƏSR TƏRTİBLİ DİFFERENSİAL MƏSƏLƏNİN İKİNCİ NÖV FREDHOLM İNTEQRAL TƏNLİYİNƏ GƏTİRİLMƏSİ

Əliyev N.A.[1], Rəsulov M.A.[2*], Cəlalov G.İ.[2], Sinsoisal B.[3], Xanbabayeva M.X.[2]

[1]Bakı Dövlət Universiteti, Tətbiqi Riyaziyyat və Kibernetika Fakültəsi, Azərbaycan
AZ1148, Bakı, akad. Zahid Xəlilov küç., 23
[2]Azərbaycan Respublikasının Elm və Təhsil Nazirliyi, Neft və Qaz İnstitutu, Azərbaycan
AZ1000, Bakı, Fikrət Əmirov küç., 9
[3]İstanbul Beykent Universiteti, Mühəndislik və Memarlıq Fakültəsi, Kompüter Mühəndisliyi Bölümü, Türkiyə
34396, Sarıyer, İstanbul
**Yazışmalara məsul:mresulov@gmail.com*

Xülasə. Son onillikdə fraktal materiallarda kütlə köçürmə nəzəriyyəsi həm praktiki, həm də nəzəri cəhətdən diqqətləri cəlb etmişdir. Bu maraq məsaməli mühitin qeyri-xəttiliyini, o cümlədən onların blok strukturu və fraktal xassələrini nümayiş etdirən geofiziki və eksperimental məlumatlara əsaslanır. Belə mühitlərdə süzülmə prosesi, hətta bircins qarışıqlar üçün də, kəsr tərtibli diferensial tənliklərlə modelləşdirilir ki, bu da nəzəri tədqiqatları çətinləşdirir. Bu işdə fraktal məsaməli mühitlərdə birfazalı mayelərin müstəvi-radial hərəkətini təsvir edən qeyri-stasionar kəsr tərtibli filtrasiya tənliyinin analitik həlli qurulmuşdur. Təzyiq funksiyasının Laplas çevirməsinin törəməsi üçün gətirilmiş spektral məsələni həll etmək üçün müntəzəm nüvəli ikinci növ Volterra inteqral tənliyi alınmışdır. Bu inteqral tənlik üçün ardıcıl əvəzetmə metodundan istifadə etməklə sürətlə yığılan sıra şəklində rezolventa tapılmışdır. Sıranın yığılması üçün zəruri şərt tapılır və naməlum təzyiq funksiyasının Laplas çevirməsinin törəməsi üçün analitik ifadə alınır. Laplas çevrilməsi üçün tapılmış analitik ifadəni fəza dəyişəninə görə inteqrallamaqla orijinal funksiyanın Laplas cevirməsi üçün inteqral təsviri tapılır. Nəhayət, tərs Laplas çevrilməsini tətbiq edərək, biz axtarılan orijinal təzyiq funksiyasının dinamik paylanması üçün ifadə alırıq. Alınan ifadə, birincisi, fraktal təbiətli radial mühitdə qeyri-sabit maye axını problemini öyrənməyə imkan verir, ikincisi, baxılan problemin mövcudluğu və yeganəlyi suallarına da cavab verir. Ədədi inteqrallama üsullarından istifadə edərək, yatağın hər hansı bir r nöqtəsində və istənilən t anında təzyiq funksiyasının qiymətlərini xətanın qiymətləndirməsi ilə hesablaya bilərik. Bundan əlavə, əldə edilən nəticələr təqribi üsullardan istifadə edilərək alınan həlləri də qiymətləndirməyə imkan verir.

Açar sözlər: *fraktal mühitdə kütlə ötürülməsi, müstəvi-radial bölgədə qeyri-stasionar diffuziya tənliyi, Laplas çevrilməsi, requlyar nüvəli ikinci növ Volterra inteqral tənliyi, ardıcıl əvəzetmə üsulu*